

Circular strings on product manifolds

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1991 J. Phys. A: Math. Gen. 24 5181

(<http://iopscience.iop.org/0305-4470/24/21/027>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.252.86.83

The article was downloaded on 01/06/2010 at 14:00

Please note that [terms and conditions apply](#).

Circular strings on product manifolds

A L Larsen

Fysisk Institut, Odense University, Odense, Denmark

Received 29 May 1991

Abstract. Classical circular string configurations of the Davidson–Wali type, which exist on the product manifold of Minkowski space and a general ‘internal’ space, are investigated. The dynamics in internal space is found to be governed by a potential of characteristic ‘attraction + repulsion’ form. A few examples are presented including the system $H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 y^2 + (1/x^2 y^2))$.

1. Introduction

A very interesting non-collapsing classical string configuration was found by Davidson and Wali in [1, 2]. They considered a circular string living on the manifolds $M_4 \otimes S_1$ and $M_4 \otimes S_2$ respectively, and found that the interplay between Minkowski space and the internal spaces led to a type of self-interaction with the string, preventing it from collapsing in real space. This interplay between Minkowski space and internal space was, in the case of $M_4 \otimes S_1$, obtained by introducing a winding number, letting the string wind around the internal space S_1 , but it was somewhat surprising that a similar construction was possible in the case of $M_4 \otimes S_2$, since no topological reason would prevent the string from shrinking to a point on S_2 .

The solution by Davidson and Wali has been generalized to the case where the internal space is a sphere of arbitrary dimension [3, 4], and in the present article we consider a generalization to a much broader class of manifolds. In section 2 we define this class of manifolds and we derive the corresponding equations of motion for the string. It turns out that they are in fact of the same general form as in the original $M_4 \otimes S_2$ case [2]. In section 3 we have a few comments on the general results, and some simple examples are briefly examined.

2. The equations of motion

We consider a Nambu–Goto string which exists on a manifold which is the product of Minkowski space M_4 and an n -dimensional space, which we call X_n . Minkowski space is parametrized using cylinder coordinates in 3-space, so that the line element is

$$ds_{M_4}^2 = -dt^2 + dz^2 + dr^2 + r^2 d\theta^2. \quad (1)$$

The n -dimensional space X_n is parametrized by coordinates (x^1, x^2, \dots, x^n) so that

$$ds_{X_n}^2 = g_{ij} dx^i dx^j \quad (2)$$

where

$$g_{ij} = g_{ij}(x^1, x^2, \dots, x^n) \tag{3}$$

and $i, j = 1, 2, \dots, n$.

We now make a few assumptions about the space X_n . First we assume that we can find a coordinate system where the metric is globally independent of some of the coordinates:

$$x^i \equiv (x^\alpha, x^\mu) \quad g_{ij} = g_{ij}(x^\alpha) \tag{4}$$

where $\alpha = 1, 2, \dots, k < n$ and $\mu = (k + 1), (k + 2), \dots, n$. Furthermore we assume that the metric can be written in a direct product form:

$$g_{ij} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix}. \tag{5}$$

The assumptions (4) and (5) are a restriction which, however, seems to cover a lot of interesting spaces including spheres, ellipsoids, hyperboloids, paraboloids, cones and tori of various dimensions.

The Nambu–Goto string is defined by the Lagrangian

$$\mathcal{L} = \sqrt{-\det \mathbf{G}} \tag{6}$$

where \mathbf{G} is the induced metric on the worldsheet

$$G_{ab} = g_{MN} \frac{\partial x^M}{\partial \xi^a} \frac{\partial x^N}{\partial \xi^b} \tag{7}$$

and where g_{MN} is the metric from (1) and (2) (M, N taking $4 + n$ values), ξ^a are the usual string coordinates (τ, σ) and x^M are coordinates in $M_4 \otimes X_n$, i.e. $x^M = (t, z, r, \theta, x^\alpha, x^\mu)$. We make the following ansatz for the (τ, σ) dependence of the coordinates:

$$r = r(\tau) \quad z = 0 \quad \theta = \sigma \quad t = \tau \quad x^\alpha = A^\alpha(\tau) \quad x^\mu = B^\mu(\tau) + n^\mu \sigma \tag{8}$$

where $(n^{k+1}, n^{k+2}, \dots, n^n)$ are S_1 winding numbers ($n^\mu \in \mathbb{Z}$), so we should think of x^α as ‘radial’ coordinates and x^μ as ‘angular’ coordinates in the internal space. From (7) and (8) it follows that

$$\begin{aligned} G_{00} &= \dot{r}^2 - 1 + g_{\alpha\beta} \dot{A}^\alpha \dot{A}^\beta + g_{\mu\nu} \dot{B}^\mu \dot{B}^\nu \\ G_{11} &= r^2 + g_{\mu\nu} n^\mu n^\nu \\ G_{01} &= G_{10} = g_{\mu\nu} n^\nu \dot{B}^\mu. \end{aligned} \tag{9}$$

Using (6) and (9) we can now derive the equations of motion. The t, θ equations lead to

$$\begin{aligned} G_{11}/\mathcal{L} &= 1/\omega \\ r^2 G_{01}/\mathcal{L} &= L \\ G_{00}/\mathcal{L} &= \omega((L^2/r^4) - 1) \end{aligned} \tag{10}$$

where $1/\omega$ and L are the constant energy density and angular momentum density of the string respectively. Using (10) the r equation is

$$\frac{\ddot{r}}{\omega^2} + r \left(1 - \frac{L^2}{r^4} \right) = 0 \quad (11)$$

which can be integrated to

$$\frac{\dot{r}^2}{\omega^2} + r^2 + \frac{L^2}{r^2} = c_1^2 \quad (12)$$

and finally solved by

$$r^2(\tau) = \frac{1}{2}c_1^2 + \sqrt{\frac{1}{4}c_1^4 - L^2} \cos(2\omega\tau + \text{constant}). \quad (13)$$

The equations of motion associated with the cyclic coordinates x^μ lead to $n - k$ constant charge densities Ω_μ (also using (10)):

$$g_{\mu\nu} \left(\frac{\dot{B}^\nu}{\omega} - \frac{Ln^\nu}{r^2} \right) = \Omega_\mu \quad (14)$$

i.e.

$$\dot{B}^\mu = \omega \left(\Omega^\mu + \frac{Ln^\mu}{r^2} \right) \quad (15)$$

where

$$\Omega^\mu = \Omega^\mu(A^\alpha) = g^{\mu\nu}\Omega_\nu. \quad (16)$$

Supposing A^α is known we then obtain B^μ simply by integration of (15). It follows that the 'interesting' field equations are the equations associated with A^α , which we now consider. Using (10) and (15) we find

$$\frac{1}{\omega^2} \left[g_{\alpha\beta} \ddot{A}^\beta + \frac{\partial g_{\alpha\beta}}{\partial A^\gamma} \dot{A}^\gamma \dot{A}^\beta - \frac{1}{2} \frac{\partial g_{\gamma\beta}}{\partial A^\alpha} \dot{A}^\gamma \dot{A}^\beta \right] + \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial A^\alpha} (n^\mu n^\nu - \Omega^\mu \Omega^\nu) = 0. \quad (17)$$

Introducing the Christoffel symbols:

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial A^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial A^\beta} - \frac{\partial g_{\beta\gamma}}{\partial A^\delta} \right) \quad (18)$$

we can rewrite (17) as

$$\ddot{A}^\alpha + \Gamma_{\beta\gamma}^\alpha \dot{A}^\beta \dot{A}^\gamma + \frac{\omega^2}{2} g^{\alpha\beta} \frac{\partial g_{\mu\nu}}{\partial A^\beta} (n^\mu n^\nu - \Omega^\mu \Omega^\nu) = 0. \quad (19)$$

It is now important to realize that (19) are precisely the Hamilton equations which can be determined from the effective Hamiltonian:

$$H(A^\alpha, \dot{A}^\alpha) = \frac{1}{2\omega^2} g_{\alpha\beta} \dot{A}^\alpha \dot{A}^\beta + \frac{1}{2} g_{\mu\nu} (n^\mu n^\nu + \Omega^\mu \Omega^\nu) \equiv \frac{c_2^2}{2} \quad (20)$$

where $c_2^2/2$ is the constant value of the 'energy'. The equations of motion have now been separated into three families (12), (15) and (20), which are of the same general form as in the original $M_4 \otimes S_2$ case [2]. Finally when confronting these equations with the t, θ equations (10), we get the consistency conditions among the various constants of integration:

$$c_1^2 + c_2^2 = 1/\omega^2 \quad (21)$$

$$L = n^\mu \Omega_\mu. \quad (22)$$

3. Comments and examples

It is seen from (13) that the dynamics of the string in Minkowski space only depends on the internal space X_n through the consistency conditions among the constants of integration (21) and (22), and it is actually the winding numbers and charges from X_n that prevent the string from collapsing in Minkowski space. It is clear from (12) that a non-zero angular momentum L will ensure a strictly positive string radius. According to (22) we then just have to choose the charges Ω_μ and the winding numbers n^μ such that the combination $n^\mu \Omega_\mu$ is non-zero to get a non-collapsing string. An interesting feature is that our ansatz configuration (8) winding around the internal space X_n seems to be meaningful in many cases, even when the fundamental group of X_n is trivial and where no topological reasons prevent the string from shrinking to a point. This was first pointed out in [2] for the case $X_n = S_2$ and later observed for a sphere of arbitrary dimension [3, 4]. The origin of this phenomenon seems to be the general form of the potential in (20), which in terms of the constant charges Ω_μ can be written as

$$V(A^\alpha) = \frac{1}{2} g_{\mu\nu} n^\mu n^\nu + \frac{1}{2} g^{\mu\nu} \Omega_\mu \Omega_\nu. \quad (23)$$

Since $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$ it follows that an eventual attractive component, trying to shrink the string, in the $g_{\mu\nu} n^\mu n^\nu$ term is always accompanied by a repulsive component in the $g^{\mu\nu} \Omega_\mu \Omega_\nu$ term and vice versa. This can be illustrated by the following simple example.

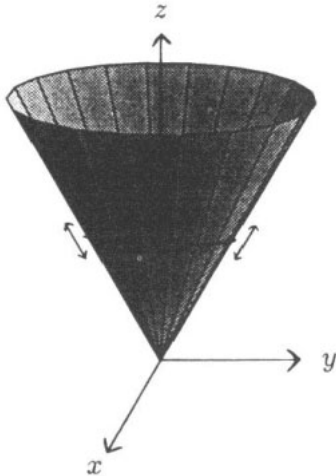


Figure 1. The self-interacting string winding around the cone is kept away from the singular point $z = 0$.

We take X_n to be a two-dimensional cone defined by (figure 1)

$$x^2 + y^2 = z^2 \quad z \geq 0 \quad (24)$$

which can be parametrized by

$$\begin{aligned} x &= z \cos \varphi & \varphi &\in [0, 2\pi] \\ y &= z \sin \varphi & z &\in [0, \infty]. \end{aligned} \quad (25)$$

The line element is

$$ds^2 = 2dz^2 + z^2 d\varphi^2 \tag{26}$$

and the metric has the desired form (4),(5) with $(x^\alpha, x^\mu) = (z, \varphi)$. Using ansatz (8) with z as the A -coordinate and φ as the B -coordinate we find the Hamiltonian (20) determining the time dependence of the z -coordinate:

$$H = \frac{1}{2}(2\dot{z}^2 + z^2 n^2 + \Omega^2 / z^2) \equiv \frac{1}{2}c_2^2 \tag{27}$$

where ω has been absorbed in τ . It is now clear from (27) that, because of the charge Ω , the string is always kept away from the singular point $z = 0$ and is therefore not allowed to shrink to a point. This is also manifest in the exact solution:

$$z^2(\tau) = \frac{c_2^2}{2n^2} + \frac{1}{n} \sqrt{\frac{c_2^4}{4n^2} - \Omega^2 \cos(\sqrt{2}n\tau + \text{constant})}. \tag{28}$$

To demonstrate the use of formula (20) properly we should consider a more complicated internal manifold than a two-dimensional cone. Take for instance a four-dimensional torus $S_3 \otimes S_1$ defined by

$$(\sqrt{x^2 + y^2 + z^2 + v^2} - a)^2 + w^2 = 1 \tag{29}$$

where a is the constant radius of $S_3(a > 1)$. The torus can be parametrized in the following way:

$$\begin{aligned} x &= (a + \cos \psi)\rho \cos \varphi_1 \\ y &= (a + \cos \psi)\rho \sin \varphi_1 \\ z &= (a + \cos \psi)\sqrt{1 - \rho^2} \cos \varphi_2 \\ v &= (a + \cos \psi)\sqrt{1 - \rho^2} \sin \varphi_2 \\ w &= \sin \psi \end{aligned} \tag{30}$$

where $\varphi_1 \in [0, 2\pi], \varphi_2 \in [0, 2\pi], \psi \in [0, 2\pi]$ and $\rho \in [0, 1]$. The line element is given by

$$ds_{S_3 \otimes S_1}^2 = d\psi^2 + (a + \cos \psi)^2 \left[\frac{d\rho^2}{1 - \rho^2} + \rho^2 d\varphi_1^2 + (1 - \rho^2) d\varphi_2^2 \right] \tag{31}$$

and it follows that the metric has the desired form (4) and (5) with $x^\alpha = (\rho, \psi)$, $x^\mu = (\varphi_1, \varphi_2)$. For the ansatz (8) we can now without further calculations write down the Hamiltonian (20) for $\rho(\tau), \psi(\tau)$, which through (19), (15) and (13) completely determines the dynamics of the string on $M_4 \otimes S_3 \otimes S_1$: (for simplicity we take $(n^1)^2 = (n^2)^2 \equiv n^2$)

$$\begin{aligned} H &= \frac{1}{2} \left[(a + \cos \psi)^2 \frac{\dot{\rho}^2}{1 - \rho^2} + \dot{\psi}^2 \right] + \frac{n^2}{2} (a + \cos \psi)^2 \\ &\quad + \frac{1}{2(a + \cos \psi)^2} \left(\frac{\Omega_1^2}{\rho^2} + \frac{\Omega_2^2}{1 - \rho^2} \right) \\ &= \frac{c_2^2}{2}. \end{aligned} \tag{32}$$

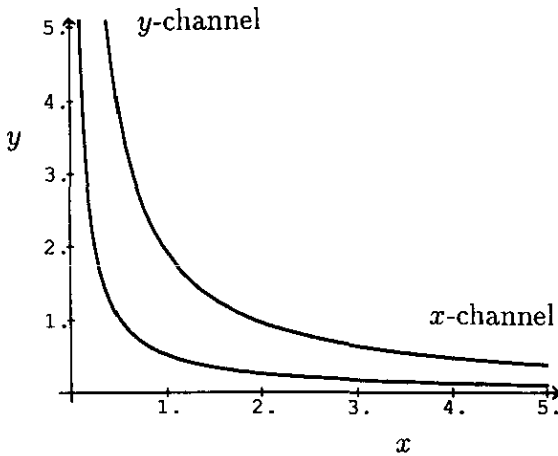


Figure 2. The potential contour of (35).

This is a two-dimensional integrable Hamiltonian. The second integral is given by

$$I_2 = (a + \cos \psi)^4 \frac{\dot{\rho}^2}{1 - \rho^2} + \frac{\Omega_1^2}{\rho^2} + \frac{\Omega_2^2}{1 - \rho^2} \equiv c^2 \quad (33)$$

and the coordinates can be separated by elimination of $\dot{\rho}^2$ in (32):

$$c^2 = \dot{\psi}^2 + n^2(a + \cos \psi)^2 + c^2 / (a + \cos \psi)^2 \quad (34)$$

where c^2 is the constant value of I_2 .

The final example we will consider here is based upon the fact that the kinetic energy of (20) only involves $g_{\alpha\beta}$, whereas the potential energy only involves $g_{\mu\nu}$. This means that we can to some extent circumvent the whole situation: instead of determining the Hamiltonian (20) for a given manifold X_n , we can choose a Hamiltonian and then try to fix $g_{\alpha\beta}$ and $g_{\mu\nu}$, to bring the kinetic and potential energy, respectively, in the form of (20). We use this procedure to construct a simple metric leading to chaotic motion. To be more specific we consider the following two-dimensional Hamiltonian:

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 y^2 + 1/x^2 y^2) \quad (35)$$

which is of the form (20) if we make the identifications

$$A^1 = x \quad A^2 = y \quad g_{\mu\nu} = x^2 y^2 \quad g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (36)$$

and where the charge and winding number for simplicity have been taken equal to unity. The potential of (35) is illustrated in figure 2. *A priori* it could be expected that the trajectory would simply escape along one of the 'channels', since the energy obviously does not prevent that. However, numerical investigations show that this does not happen (figure 3), instead one finds that the trajectory somehow 'oscillates' between the two channels. This phenomenon can be very easily understood by the principle of adiabatic invariance, as we show in the appendix.

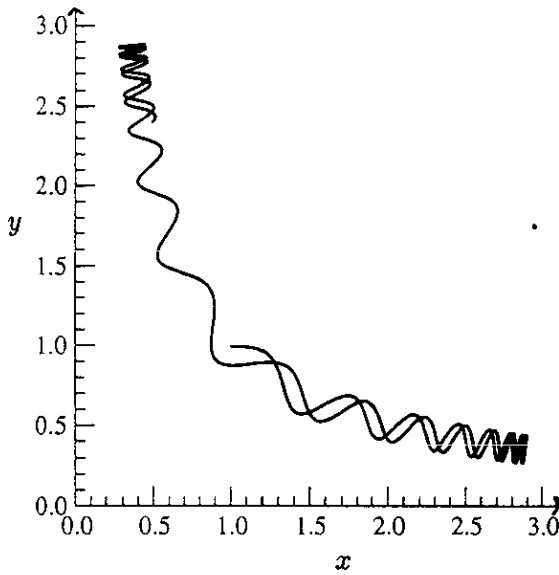


Figure 3. The trajectory of (35).

We now turn to the numerical investigations of the system defined by (35). We rotate the (x, y) axes by 45° ; then the Hamiltonian is

$$H(\xi, \eta, \dot{\xi}, \dot{\eta}) = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) + \frac{1}{8}(\xi^2 - \eta^2)^2 + 2/(\xi^2 - \eta^2)^2 \quad (37)$$

where $\xi = (x - y)/\sqrt{2}$ and $\eta = (x + y)/\sqrt{2}$. Using this parametrization we consider the surface of section defined by

$$\xi = 0 \quad \dot{\xi} > 0. \quad (38)$$

The numerical integrations are performed using the fourth-order Runge–Kutta technique; they are somewhat difficult to carry out since the trajectory spends most of the time in the channels. The minimum energy of the Hamiltonian (37) is $E = 1$, and we have studied the surface of section (38) for a large number of initial conditions in the energy range $E \in [1; 2]$. The general macroscopic picture of these investigations is the following: For small energies close to $E = 1$ the points seem to lie exactly on a curve (figure 4(a)), thus indicating regular motion. However, when the energy is raised, these curves become more and more smeared out (figures 4(b) and 4(c)), and when the energy approaches $E = 2$ the surface of sections are dominated by large irregular regions (figure 4(d)), and it is impossible to draw a curve through the points. This is very similar to the results obtained for other non-scale invariant Hamiltonians, for instance the Hénon–Heiles system [5] and some systems with quartic potentials [7], and it strongly indicates that no second integral of motion exists for the system (35), i.e. it is non-integrable. It should be stressed here that in the three examples given earlier, we have only considered the motion governed by the Hamiltonian (20) in internal space. In all three cases there is also an equation (12) for the string radius in Minkowski space and equations (15) for the angular coordinates in internal space. The examples have shown that we can obtain both regular and irregular motion in internal space, but from the four-dimensional point of view the string is ‘breathing’

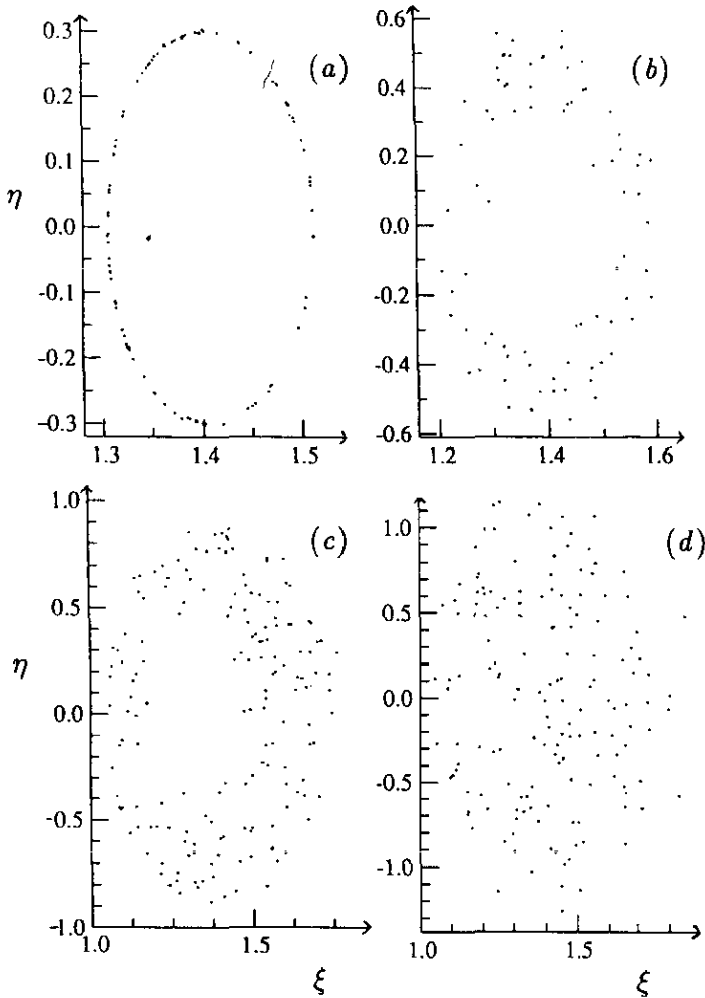


Figure 4. The surface of section (38) for various energies of the Hamiltonian system (37). (a), (b), (c) and (d) represent the surface of section at $E = 1.09$, $E = 1.28$, $E = 1.44$ and $E = 1.85$ respectively.

completely regularly in all cases since, as pointed out at the beginning of this section, it only seems to feel the constant charges and winding numbers from internal space, which exist independently of the nature of the internal motion.

In conclusion we have studied a circular string which exists on the product manifold of Minkowski space M_4 and a general internal space X_n , thus generalizing the results of [1-4]. We found that the motion in internal space was governed by a potential of characteristic 'attraction + repulsion' form. A simple example $V(x, y) = \frac{1}{2}(x^2 y^2 + 1/x^2 y^2)$ was examined numerically, giving strong indications for irregular motion, although it should be noted that our numerical studies have been somewhat cursory; a more systematic investigation concerning regular and irregular regions is, however, beyond the scope of this article.

Acknowledgments

I would like to thank N K Nielsen for a careful reading of the manuscript, and L Lolle, H G Kjærgaard and J Jørgensen for help in the computer work.

Appendix

In this appendix we show that the trajectory of the Hamiltonian system (35) is always forced to go back somewhere in the channels of the potential (figure 2), even though the energy does not prevent it from escaping to infinity. Suppose for instance that the trajectory is somewhere in the x -channel. The motion in the y -direction is then governed by the Hamiltonian:

$$H_y(y, \dot{y}) = \frac{1}{2}(\dot{y}^2 + x^2 y^2 + 1/x^2 y^2). \quad (\text{A1})$$

If we consider x as a parameter we can solve this Hamiltonian by

$$y^2(\tau) = \frac{1}{x^2} [\sqrt{E_y - 1} \sin(2x\tau + \text{constant}) + E_y] \quad (\text{A2})$$

where E_y is the value of H_y , so in the Hamiltonian represents a (non-linear) oscillator of frequency $2x$. Still keeping x fixed we can calculate the constant action variable J_y corresponding to y :

$$\begin{aligned} J_y &\equiv \frac{1}{2\pi} \oint \dot{y} dy \\ &= \frac{1}{2\pi} \oint \sqrt{2E_y - x^2 y^2 - 1/x^2 y^2} dy \\ &= (E_y - 1)/2x. \end{aligned} \quad (\text{A3})$$

According to the adiabatic invariance theorem [6, 7] (see also [8, 9] for applications) the action (A3) stays constant under small variations of the frequency $2x$, and that is exactly the kind of situation we are in far out in the x -channel, where the relative change in x is small over a period of y . In this approximation we then get from (A1) and (A3):

$$H \approx \frac{1}{2} \dot{x}^2 + 2xJ_y + 1. \quad (\text{A4})$$

It follows that $\dot{x} = 0$ when

$$x = (E - 1)/2J_y \quad (\text{A5})$$

where E is the constant value of the total energy. When x reaches this value, the trajectory is forced to return to the central region.

References

- [1] Davidson A and Wali K C 1988 *Phys. Lett.* **213B** 439
- [2] Davidson A and Wali K C 1988 *Phys. Rev. Lett.* **248B** 1450
- [3] Djurhuus J, Larsen A L and Nielsen M K 1990 *Phys. Lett.* **248B** 273
- [4] Larsen A L 1991 *Europhys. Lett.* **14** 733
- [5] Hénon M and Heiles C 1964 *Astronom. J.* **69** 73
- [6] Goldstein H 1980 *Classical Mechanics* 2nd edn (New York: Addison-Wesley) p 531
- [7] Arnold V I 1963 *Russ. Math. Surveys.* **18** 85
- [8] Carnegie A and Percival I C 1984 *J. Phys. A: Math. Gen.* **17** 801
- [9] Chang S 1984 *Phys. Rev. D* **29** 259